Correction Model Final Exam LA2, 2021

1) a) $S \subset C[-1,1]$, $S = \text{span}(1, x, x^2)$.

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\,dx$$

$$\|1\|^2 = \int_{-1}^{1} 1\,dx = [x]_{-1}^{1} = 2 \Rightarrow \|1\| = \sqrt{2}$$

Take $u_1(x) = \frac{1}{\sqrt{2}}$ as the first basis vector

$$p_1(x) = \langle x, \tfrac{1}{\sqrt{2}}\rangle \cdot \tfrac{1}{\sqrt{2}}$$

$$\frac{1}{\sqrt{2}}\int_{-1}^{1} x\,dx = \frac{1}{\sqrt{2}}\left[\frac{1}{2}x^2\right]_{-1}^{1} = 0$$

Hence $p_1(x) = 0$. Take $u_2(x) = \frac{x}{\|x\|}$

Compute $\|x\|^2 = \int_{-1}^{1} x^2\,dx = \left[\frac{1}{3}x^3\right]_{-1}^{1} = \frac{2}{3}$

so $\|x\| = \sqrt{2}/\sqrt{3}$.

Second basis vector becomes $u_2(x) = \frac{\sqrt{3}\,x}{\sqrt{2}}$

Next: $p_2(x) = \langle x^2, \frac{1}{\sqrt{2}}\rangle \cdot \frac{1}{\sqrt{2}} + \langle x^2, \frac{\sqrt{3}}{\sqrt{2}}x\rangle \cdot \frac{\sqrt{3}}{\sqrt{2}}x$

$$\frac{1}{\sqrt{2}}\int_{-1}^{1} x^2\,dx = \frac{1}{\sqrt{2}}\left[\frac{1}{3}x^3\right]_{-1}^{1} = \frac{1}{\sqrt{2}}\,\frac{2}{3} = \frac{2}{3\sqrt{2}}$$

$$\frac{\sqrt{3}}{\sqrt{2}}\int_{-1}^{1} x^3\,dx = \frac{\sqrt{3}}{\sqrt{2}}\left[\frac{1}{4}x^4\right]_{-1}^{1} = 0$$

Hence: $p_2(x) = \frac{2}{3\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = \frac{1}{3}$.

So $u_3(x) = \dfrac{x^2 - \frac{1}{3}}{\| x^2 - \frac{1}{3} \|}$

Compute $\| x^2 - \frac{1}{3} \|^2 = \int_{-1}^{1} (x^2 - \frac{1}{3})^2 \, dx$

$= \int_{-1}^{1} x^4 - \frac{2}{3}x^2 + \frac{1}{9} \, dx = \left[\frac{1}{5}x^5 - \frac{2}{9}x^3 + \frac{1}{9}x \right]_{-1}^{1}$

$= \left(\frac{1}{5} - \frac{2}{9} + \frac{1}{9} \right) - \left(-\frac{1}{5} + \frac{2}{9} - \frac{1}{9} \right)$

$= \frac{2}{5} - \frac{4}{9} + \frac{2}{9} = \frac{2}{5} - \frac{2}{9} = \frac{18}{45} - \frac{10}{45} = \frac{8}{45}$

So $\| x^2 - \frac{1}{3} \| = \dfrac{\sqrt{8}}{\sqrt{45}} = \dfrac{2\sqrt{2}}{3\sqrt{5}}$

Third basis vector : $u_3(x) = \dfrac{3\sqrt{5}}{2\sqrt{2}} (x^2 - \frac{1}{3})$

b) Projection of x^3 onto S :

$p(x) = \langle x^3, \frac{1}{\sqrt{2}} \rangle \cdot \frac{1}{\sqrt{2}} + \langle x^3, \frac{\sqrt{3}x}{\sqrt{2}} \rangle \frac{\sqrt{3}x}{\sqrt{2}}$

$+ \langle x^3, \frac{3\sqrt{5}}{2\sqrt{2}}(x^2 - \frac{1}{3}) \rangle \cdot \frac{3\sqrt{5}}{2\sqrt{2}}(x^2 - \frac{1}{3})$

$\frac{1}{\sqrt{2}} \int_{-1}^{1} x^3 \, dx = \frac{1}{\sqrt{2}} \left[\frac{1}{4} x^4 \right]_{-1}^{1} = 0$

$\frac{\sqrt{3}}{\sqrt{2}} \int_{-1}^{1} x^4 \, dx = \frac{\sqrt{3}}{\sqrt{2}} \left[\frac{1}{5} x^5 \right]_{-1}^{1} = \frac{\sqrt{6}}{5}$

$$\frac{3\sqrt{5}}{2\sqrt{2}} \int_{-1}^{1} x^3\left(x^2 - \tfrac{1}{3}\right) dx = 0$$

odd function !

So

$$p(x) = \frac{\sqrt{6}}{5} \frac{\sqrt{3}}{\sqrt{2}} x = \frac{3}{5} x$$

2) a) J is in Jordan form, so is upper triangular
The product of finitely many upper triangular matrices is again upper triangular.
Clearly, the diagonal elements of J^k are $\lambda_1^k, \lambda_2^k, \ldots, \lambda_n^k$

(Note: the above answer is sufficient, I do not expect "long proofs")

b) By a), $g(J)$ is the sum of upper triangular matrices, so $g(J)$ is upper triangular. Also, the diagonal elements of $g(J)$ are $g(\lambda_1), g(\lambda_2), \ldots, g(\lambda_n)$
The conclusion is that these are also the eigenvalues of $g(J)$

c) I will show that $\underline{g}(A)$ and $\underline{g}(\mathcal{J})$ are similar. Indeed, we know that $S^{-1}AS=\mathcal{J}$ for some nonsingular matrix S.
Let $\underline{g}(s) = g_k s^k + \cdots g_1 s + g_0$, $g_i \in \mathbb{C}$.
Then

$$\underline{g}(\mathcal{J}) = g_k \mathcal{J}^k + \cdots + g_1 \mathcal{J} + g_0 I$$

$$= g_k (S^{-1}A\ S)^k + \cdots + g_1 S^{-1}AS + g_0 I$$

Note that $(S^{-1}AS)^l = S^{-1}A^l S$ for any l
Thus

$$\underline{g}(\mathcal{J}) = S^{-1}\left(g_k A^k + \cdots + g_1 A + g_0 I\right) S$$

$$= S^{-1}\underline{g}(A)\, S.$$

Conclusion: $\underline{g}(\mathcal{J})$ and $\underline{g}(A)$ are similar, and therefore have the same eigenvalues, which, by b), are $\underline{g}(\lambda_1) \cdots \underline{g}(\lambda_n)$.

(This proves the so-called "spectral mapping theorem": the eigenvalues of A "go" where the polynomial goes.)

3) $A \in \mathbb{R}^{n\times n}$, $A^T X + XA < 0$ is a linear matrix inequality in the unknown symmetric $X \in \mathbb{R}^{n\times n}$

a) Let $\lambda \in \mathbb{C}$ be an eigenvalue of A. Let $v \neq 0$ $v \in \mathbb{C}^n$ be an eigenvector, i.e. $Av = \lambda v$.
Note that $v^H A^T = (Av)^H = (\lambda v)^H = v^H \bar{\lambda}$
Now let $X > 0$ be a solution to

$$A^T X + XA < 0$$

Then

$$v^H (A^T X + XA) v < 0 \quad \text{(scalar!)}$$

Hence $\quad v^H A^T X v + v^H X A v < 0$, so

$$\bar{\lambda}\, v^H X v + v^H X \lambda v < 0$$

This implies

$$(\lambda + \bar{\lambda})\, v^H X v < 0$$

so

$$2\,\mathrm{Re}(\lambda) \,.\, v^H X v < 0$$

Since $v^H X v > 0$, this implies $\mathrm{Re}(\lambda) < 0$.

b) If $\mathrm{Re}(\lambda) < 0$ then $\int_0^\infty e^{A^T t} e^{At}\, dt$ converges.
Let $X := a \int_0^\infty e^{A^T t} e^{At}\, dt$.

Then $A^T X + XA =$

$$a A^T \int_0^\infty e^{A^T t} e^{At} \, dt + a \int_0^\infty e^{A^T t} e^{At} \, dt \, . \, A =$$

$$a \int_0^\infty A^T e^{A^T t} e^{At} + e^{A^T t} e^{At} A \, dt =$$

$$a \int_0^\infty \frac{d}{dt} \left(e^{A^T t} e^{At} \right) dt =$$

$$a \left[e^{A^T t} e^{At} \right]_0^\infty = -aI$$

c) Let $v \neq 0$. Then

$$v^T X v = a \int_0^\infty v^T e^{A^T t} e^{At} v \, dt$$

$$= a \int_0^\infty \| e^{At} v \|^2 \, dt$$

Since $e^{At} v \neq 0$ for all t, we have that $\| e^{At} v \|^2 > 0$ for all t, so $\int_0^\infty \| e^{At} v \|^2 dt > 0$ as well. Since $a > 0$, we find $v^T X v > 0$ so $X > 0$.

d) If $Re(\lambda) < 0$ for all eigenvalues of A we know that $(*)$ is a solution to $A^T X + XA = -aI$. This implies $A^T X + XA < 0$.

Conversely, in a) we showed that if $A^TX+XA<0$ has a solution $X>0$ then all eigenvalues λ of A satisfy $Re(\lambda)<0$.

4.) a) $u_1 = \begin{pmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ \frac{2}{3} \end{pmatrix}$. Clearly, $\|u_1\| = 1$.

In order to find u_2 and u_3 so that $U=(u_1\ u_2\ u_3)$ is an orthogonal matrix we apply Gram Schmidt to the vectors $\{u_1, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\}$, which clearly form a basis of $\mathbb{R}^3$.

So: u_1 we take as first vector. Then:

$$P_1 = \langle \begin{pmatrix}1\\0\\0\end{pmatrix}, u_1\rangle \cdot u_1 = \begin{pmatrix} 1/9 \\ -2/9 \\ 2/9 \end{pmatrix}$$

Then:

$$u_2 := \frac{\begin{pmatrix}1\\0\\0\end{pmatrix} - \begin{pmatrix}1/9\\-2/9\\2/9\end{pmatrix}}{\left\|\begin{pmatrix}1\\0\\0\end{pmatrix} - \begin{pmatrix}1/9\\-2/9\\2/9\end{pmatrix}\right\|} \qquad \begin{pmatrix}8/9\\2/9\\-2/9\end{pmatrix}$$

$$\left(\frac{64}{81} + \frac{4}{81} + \frac{4}{81}\right)^{1/2} = \sqrt{\frac{72}{81}}$$

$$= \frac{\sqrt{9\cdot 8}}{9} = \frac{1}{3}\sqrt{8} = \frac{2}{3}\sqrt{2}$$

So

$$u_2 = \frac{3}{2\sqrt{2}} \begin{pmatrix}8/9\\2/9\\-2/9\end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix}4/3\\1/3\\-1/3\end{pmatrix}$$

$$p_2 = \left\langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, u_1 \right\rangle \cdot u_1 + \left\langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, u_2 \right\rangle \cdot u_2$$

$$= -\frac{2}{3} u_1 + \frac{1}{3\sqrt{2}} u_2$$

$$= -\frac{2}{3} \begin{pmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ \frac{2}{3} \end{pmatrix} + \frac{1}{3\sqrt{2}} \frac{1}{\sqrt{2}} \begin{pmatrix} 4/3 \\ 1/3 \\ -1/3 \end{pmatrix}$$

$$= \begin{pmatrix} -2/9 \\ 4/9 \\ -4/9 \end{pmatrix} + \begin{pmatrix} 4/18 \\ 1/18 \\ -1/18 \end{pmatrix} = \begin{pmatrix} 0 \\ 1/2 \\ -1/2 \end{pmatrix}$$

Then

$$u_3 := \frac{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1/2 \\ -1/2 \end{pmatrix}}{\left\| \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1/2 \\ -1/2 \end{pmatrix} \right\|}$$

$$\to \begin{pmatrix} 0 \\ 1/2 \\ 1/2 \end{pmatrix}$$

$$\to \left(1/4 + 1/4 \right)^{1/2} = \frac{1}{\sqrt{2}}$$

Dus $$u_3 = \sqrt{2} \begin{pmatrix} 0 \\ 1/2 \\ 1/2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{pmatrix}$$

b) $A = \begin{pmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{pmatrix}$ $\quad A^T A = \begin{pmatrix} 9 & -9 \\ -9 & 9 \end{pmatrix}$

Eigenvalues of $A^T A$: $(9-\lambda)^2 - 81 = 0$

$\Leftrightarrow \lambda^2 - 18\lambda = 0 \Leftrightarrow \lambda(\lambda - 18) = 0$

So $\lambda_1 = 18$, $\lambda_2 = 0$.

The singular values of A are :

$$\sigma_1 = \sqrt{\lambda_1} = 3\sqrt{2}$$

$$\sigma_2 = \sqrt{\lambda_2} = 0$$

Eigenvectors : $v_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$, $v_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

Now we need to find $U = (u_1 \ u_2 \ u_3)$ so that U is <u>orthogonal</u> and

$$\begin{pmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = (u_1 \ u_2 \ u_3) \begin{pmatrix} 3\sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

This requires

$$\begin{pmatrix} \frac{2}{\sqrt{2}} \\ -\frac{4}{\sqrt{2}} \\ \frac{4}{\sqrt{2}} \end{pmatrix} = 3\sqrt{2}\, u_1$$